Homework #7 Due Friday, March 29

**Problem 1:** Two people of equal height—Alice and Bob—are carrying a uniform rectangular bookcase of mass $M$ and horizontal length $L$ at constant speed parallel to the ground in the presence of a downward gravitational acceleration $g$.

(a) If Alice is a given horizontal distance $L_A \leq L/2$ forward from the midpoint of the bookcase and Bob is a given horizontal distance $L_B \leq L/2$ backward from the midpoint of the bookcase, what upward forces $F_A$ and $F_B$ are they each exerting on the bookcase? Fully simplify your answers and express them solely in terms of $M$, $g$, $L_A$, and $L_B$. Verify that your answers make physical sense by using dimensional analysis as well as by studying the separate limiting cases of large $M$, large $g$, and $L_A \to L_B$.

(b) Given a fixed value of $L_B$ for Bob, at which distance $L_A$ should Alice stand to minimize the necessary upward force with which she must hold her side of the bookcase?

**Problem 2:** In this problem, we study the behavior of a yo-yo sitting at rest on a table as we pull its string either straight upward, or in the horizontal direction, or at certain angles between those two directions.

We regard the yo-yo as consisting of an outer cylinder of radius $R$ and an inner cylinder of radius $r < R$, where the inner cylinder is attached to a long, very thin, lightweight string and where the outer and inner cylinders are fixed permanently to each other and always rotate together.
(a) Ignoring the inner cylinder for now, we regard the yo-yo as being just a uniform cylinder of radius $R$, thickness $L$, total mass $M$, and mass per unit volume

$$\rho \equiv \frac{\text{(total mass)}}{\text{(total volume)}} = \frac{M}{\pi R^2 L} = \text{const.}$$

Compute the yo-yo’s rotational inertia $I$ around the rotation axis through its center using the definition

$$I \equiv \int dm r^2_\perp,$$

where the integration is over all the constituent elements of mass $dm$ that make up the yo-yo and $r_\perp$ is the perpendicular distance of a given mass element from the yo-yo’s central axis. Express your answer in terms of $M$ and $R$, and assume for the rest of the problem that this value of $I$ is the yo-yo’s rotational inertia.

**Hint:** Regard the infinitesimal mass element $dm$ as being a tiny “cube” of volume $d(Volume)$ according to

$$dm = \rho \times d(Volume) = \rho \times d(\text{radial thickness}) \times d(\text{arc length}) \times d(\text{thickness along axis}) = \rho \times (dr_\perp) \times (r_\perp d\theta) \times (dz),$$

and then integrate first over $0 \leq r_\perp \leq R$, treating $\theta$ and $z$ as constants, then integrate over $0 \leq \theta < 2\pi$, treating $z$ as a constant, and then finally integrate over $0 \leq z \leq L$. Your answer should take the form $I = (\text{simple fraction}) \times MR^2$.

(b) Assuming that all motion is in the two-dimensional $x, y$ plane, suppose that the yo-yo sits initially at rest on a horizontal table along the $x$ axis and that we pull the string with a given force $F_{\text{string}}$ straight upward in the positive $y$ direction from the right-hand side (that is, the positive $x$ side) of the yo-yo. Assuming that the string force acts on the inner cylinder at its point of contact with the inner cylinder in a direction tangent to the inner cylinder, and that the yo-yo can move only by rolling along the $x$ direction without slipping, determine the magnitude and direction of the yo-yo’s acceleration $a$ in terms of $F_{\text{string}}$, $M$, $R$, and $r$. 
**Hint:** Be sure to draw a free-body diagram showing all the force vectors and the points at which they act, be careful choosing your signs in a consistent manner, and don’t forget that a frictional force is responsible for preventing the yo-yo from slipping. Use Newton’s second law and the torque equation, and also use the no-slipping condition to relate the yo-yo’s ordinary acceleration \(a\) to its angular acceleration \(\alpha\).

(c) Now suppose that we instead pull the string with a given force \(F_{\text{string}}\) **horizontally** along the positive \(x\) direction, where the point of contact between the string and the inner cylinder is at the bottom of the inner cylinder. Again assuming that the string force is tangent to the inner cylinder and that the yo-yo starts at rest and can move only by rolling along the \(x\) direction without slipping, determine the **magnitude** and **direction** of the yo-yo’s acceleration \(a\) again in terms of \(F_{\text{string}}, M, R,\) and \(r\).

(d) If we pull the string with a given force \(F_{\text{string}}\) at an angle \(\theta\) measured upward relative to the positive \(x\) direction—that is, somewhere between the directions in parts (b) and (c)—and continue to assume that the direction of the string force is tangent to the inner cylinder, then for what angle \(\theta\) measured upward relative to the positive \(x\) direction will the yo-yo be unable to roll from rest without slipping?

**Problem 3:** From experiments, physicists knew by the late 1800s that the wavelength \(\lambda\) of light emitted by an energetically excited hydrogen gas could be computed from various heuristic, empirical formulas. Johann Jakob Balmer found the first reasonably successful example in 1885. In 1888, Johannes Rydberg wrote down a more precise formula, known today as the **Rydberg formula**

\[
\left( \frac{1}{\lambda} \right)_{n,m} = R_H \left( -\frac{1}{n^2} + \frac{1}{m^2} \right),
\]

where \(m\) and \(n\) are positive integers with \(m < n\) and where the **Rydberg constant** \(R_H\) had the empirically measured value

\[R_H \simeq 1.1 \times 10^7 \text{ m}^{-1} .\]

In 1913, Niels Bohr proposed a crude, “semiclassical” model for the hydrogen atom that allowed him to derive a formula for \(R_H\) in terms of more fundamental quantities, as we will explore in this exercise. His work earned him the Nobel Prize in Physics in 1922.

According to the **Bohr model**, one imagines that the hydrogen atom consists of a pointlike proton fixed in space and having positive electric charge

\[q_{\text{proton}} = +e \simeq +1.602 \times 10^{-19} \text{ C}\]

together with a pointlike electron of negative electric charge

\[q_e = -e \simeq -1.602 \times 10^{-19} \text{ C}\]

and mass

\[m_e \simeq 9.109 \times 10^{-31} \text{ kg}\]
traveling around the proton in a circular orbit at a constant speed. Due to the proton, the electron experiences an attractive, centripetal Coulombic force of magnitude

\[ F_C = \left| \frac{1}{4\pi\epsilon_0} \frac{\text{(proton charge)} \times \text{(electron charge)}}{r^2} \right| = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}, \]

where \( r \) denotes the radius of the electron’s circular orbit and where the permittivity of free space \( \epsilon_0 \) has the constant value:

\[ \epsilon_0 \simeq 8.854187817620 \times 10^{-12} \frac{C^2 \cdot s^2}{kg \cdot m^3}. \]

The centripetal Coulombic force on the electron is conservative and arises from a potential energy

\[ V(r) = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r}. \]

(a) Continuing to treat the electron like a classical particle, express its orbital speed \( v \), kinetic energy \( T \), total mechanical energy \( E = T + V \), and the magnitude \( L \) of the electron’s orbital angular momentum relative to the position of the proton all in terms of the quantities \( \epsilon_0, e, m_e, \) and \( r \).

(b) Use the electron’s orbital angular momentum \( L \) to eliminate \( r \) from the formula for the electron’s total mechanical energy \( E \), thereby expressing \( E \) in terms of \( \epsilon_0, e, m_e, \) and \( L \).

(c) Invoking the quantization of orbital angular momentum, meaning that \( L = nh = nh/2\pi \) for a positive integer \( n \) and where again \( h \) is Planck’s original constant,

\[ h \simeq 6.62606896 \times 10^{-34} J \cdot s, \]

eliminate \( L \) from your expression for \( E \) to obtain a formula of the form

\[ E \equiv E_n = -(\text{const}) \times \frac{1}{n^2}. \]

Determine the constant factor in terms of in terms of \( \epsilon_0, e, m_e, \) and \( h \), and then plug in numbers to obtain a numerical estimate in joules for the constant factor, being sure to check your units. In the present context, the integer \( n \), which labels the electron’s energy level, came to be known as the electron’s principal quantum number.

Given a quantity of electric charge carrying some overall amount of energy, the energy per unit charge is known as the corresponding voltage. Voltage can be measured in units of volts (V), where one volt is defined to be one joule per coulomb:

\[ 1 \text{ V} \equiv 1 \frac{J}{C}. \]

If we multiply a voltage by a quantity of charge, then we obtain an associated amount of energy. In particular, if we multiply one volt by the magnitude \( e \) of the charge of a single electron or proton, then we obtain a unit of energy known as the electronvolt (eV):

\[ 1 \text{ eV} = e \times 1 \text{ V} \simeq 1.6022 \times 10^{-19} \text{ J}. \]

Electronvolts are therefore a natural choice of unit for quantifying the energies of small numbers of atoms and subatomic particles.

(d) Express your constant factor from part (c) in electronvolts. You should obtain the famous result

\[ E_n \simeq -13.6 \text{ eV} \times \frac{1}{n^2}, \]

eV\times \frac{1}{n^2},

(e) An ion is an atom that doesn’t have the same number of electrons as protons and therefore has a nonzero net electric charge. Use your result from part (d) to determine the input energy \( E_{\infty} - E_1 \) needed to ionize the hydrogen atom by completely liberating its electron \( (n = 1 \rightarrow \infty) \).
For purposes of comparison, the total mass-energy \( E = mc^2 \) contained in an electron at rest according to special relativity is approximately \( 0.511 \text{ MeV} \), the mass-energy contained in a proton is \( 938 \text{ MeV} \simeq 1 \text{ GeV} \), and the mass-energy of the Higgs boson is \( 125 \text{ GeV} \), where MeV denotes a meaelectronvolt (one million electron volts) and GeV denotes a gigaelectronvolt (one billion electron volts).

(f) Setting the electron’s principal quantum number to be \( n = 1 \), calculate the smallest possible radius \( r \) of the electron’s orbit, a value known as the Bohr radius \( a_0 \) that roughly gives half the diameter of the hydrogen atom. Express your answer first in terms of \( \varepsilon_0 \), \( e \), \( m_e \), and \( h \), and then compute \( a_0 \) numerically, again being sure to check your units. According to your results, what’s the approximate diameter \( 2a_0 \) of the hydrogen atom in units of angstroms \( \text{Å} \)? (1 \( \text{Å} \equiv 10^{-10} \text{m} \).)

(g) Calculate the orbital speed \( v \) of the electron for \( n = 1 \) in meters per second, showing in particular that this speed is much less than the speed of light \( c \).

(h) Supposing for a moment that the electron classically radiates away its kinetic energy as electromagnetic radiation according to the Larmor formula,

\[
P = \frac{\text{(energy)}}{\text{(time)}} = \frac{2}{3} \frac{q^2 a^2}{4 \pi \varepsilon_0 c^3} \quad [q = e],
\]

which holds for particles moving at nonrelativistic speeds, use the approximate formula \( \tau \sim T/P \) to determine the time scale \( \tau \) over which the electron should lose all its orbital kinetic energy \( T \) and spiral into the nucleus, expressing \( \tau \) initially in terms of \( \varepsilon_0 \), \( c \), \( h \), \( m_e \), and \( e \) before calculating the result numerically in seconds to within a power of ten. (One refers to \( \tau \) as being the classical lifetime of the hydrogen atom.) Actual electrons obviously don’t behave in this way, and, indeed, the Bohr model assumes (with any real justification) that an electron in a definite orbit doesn’t radiate at all.

(i) If the electron transitions from an orbit with principal quantum number \( n \) having an energy \( E_n \) to an orbit with principal quantum number \( m \) having a lower energy \( E_m \), it emits a photon whose energy \( E_{\text{photon}} \) is precisely the energy difference

\[
(\Delta E)_{n,m} = E_n - E_m.
\]

Recalling Planck’s radiation law, we know that the energy of the photon is related to its frequency \( \nu \) by \( E_{\text{photon}} = h\nu \), and we know that the wavelength \( \lambda \) of the photon and its frequency \( \nu \) are related to the speed of light

\[
c = 299 792 458 \text{ m/s}
\]

by

\[
\lambda \nu = \text{(wavelength)} \times \text{(frequency)} = \frac{\text{(wavelength)}}{\text{(period)}} = \frac{\text{(distance covered)}}{\text{(time)}} = \text{(speed)} = c.
\]

Use these formulas to determine the inverse-wavelength \( (1/\lambda_{\text{photon}}) = (1/\lambda)_{n,m} \) of the emitted photon. Comparing your result with the empirical Rydberg formula, obtain a formula for the Rydberg constant \( R_H \) in terms of \( \varepsilon_0 \), \( c \), \( m_e \), \( h \), and \( c \), and then calculate its numerical value in inverse-meters.

It’s important not to take the Bohr model too seriously, as it deviates from our modern understanding of quantum theory in numerous ways. For one thing, quantum particles like electrons don’t follow well-defined classical trajectories, let alone circular orbits. Moreover, although angular momentum is indeed quantized, the hydrogen atom’s electron can have vanishing orbital angular momentum \( L = 0 \)!