Biomedical Informatics 260

Computational Feature Extraction: Geometric Features
Lecture 5
David Paik, PhD
Spring 2019
Correction to Last Lecture

- Fourier Transform formulas:

\[ f(x) = \int_{-\infty}^{\infty} F(u) \cdot e^{i2\pi ux} \, dx \quad \text{WRONG} \]

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) \cdot e^{i2\pi (ux + vy)} \, dx \, dy \quad \text{WRONG} \]

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) \cdot e^{i2\pi ux} \, du \, dv \quad \text{RIGHT} \]

\[ f(x) = \int_{-\infty}^{\infty} F(u) \cdot e^{i2\pi ux} \, du \quad \text{RIGHT} \]
Image Features

• Can be computed
  • Per-pixel
  • Per-object
    • Boundary
    • Region
  • Per-image

• Desirable Properties of Image Features
  • Translation, rotation, (and sometimes scale) invariance
  • Robustness to noise & acquisition protocol
  • Statistical independence from other shape features
    • Very important for machine learning
Shape Features

Geometric
- Angle
- Thickness
- Volume, Area
- Perimeter
- Surface Area
- Mean Curvature
- Gaussian Curvature
- Gradient
- Shape Index
- Curvedness
- Gaussian Curvature

Clinical
- CAD Score
- Cobb Angle, Anteversion
- Malignancy
- Cartilage Thickness
- Stenosis
- Spiculation
- Margin
- Medial Axis
- Circularity
- Tortuosity

Geometric vs. Clinical
The Shape v Texture View of the World
Local Pointwise Features
Image Partial Derivatives as Features

$0^{th}$ derivative: $I(x, y)$

$1^{st}$ derivative: $\frac{\partial}{\partial x} I(x, y)$ $|\nabla I(x, y)|$

$2^{nd}$ derivative: $\frac{\partial^2}{\partial x^2} I(x, y)$ $\frac{\partial^2}{\partial x \partial y} I(x, y)$ $\nabla^2 I(x, y)$

These analyses are often done across multiple spatial scales

Kniss et al, TVCG 2002

What causes these arch-like structures?
2D Isocontour Curvature
(of a 2D level set of an implicit function)

When $\phi$ is an implicit function, not necessarily a signed distance function:

$$\kappa = \text{div} \vec{N} = \nabla \cdot \vec{N} = \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{\partial}{\partial x} \left[ \frac{\phi_x}{(\phi_x^2 + \phi_y^2)^{1/2}} \right] + \frac{\partial}{\partial y} \left[ \frac{\phi_y}{(\phi_x^2 + \phi_y^2)^{1/2}} \right]$$

$$= \frac{\phi_y^2 \phi_{xx} - 2\phi_x \phi_y \phi_{xy} + \phi_x^2 \phi_{yy}}{(\phi_x^2 + \phi_y^2)^{3/2}}$$

Remember, subscripts are partial derivatives

Much easier for a signed distance function $S$:

$$\kappa = \nabla^2 S = \frac{\partial}{\partial x} \left[ S_x \right] + \frac{\partial}{\partial y} \left[ S_y \right]$$

$$= S_{xx} + S_{yy}$$

How can curvature be clinically useful information?
3D Isosurface Curvature
(of a 3D level set of an implicit function)

\[ \kappa_1 + \kappa_2 = \text{div} \, \vec{N} = \nabla \cdot \vec{N} = \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} \]

\[ H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2h^{3/2}} \left[ \phi_x^2 (\phi_{yy} + \phi_{zz}) - 2\phi_y \phi_z \phi_{yz} \right. \]
\[ \left. + \phi_y^2 (\phi_{xx} + \phi_{zz}) - 2\phi_x \phi_z \phi_{xz} \right. \]
\[ \left. + \phi_z^2 (\phi_{xx} + \phi_{yy}) - 2\phi_x \phi_y \phi_{xy} \right] \]

\[ K = \kappa_1 \kappa_2 = \frac{1}{h^2} \left[ \phi_x^2 (\phi_{yy} \phi_{zz} - \phi_{yz}^2) + 2\phi_y \phi_z (\phi_{xz} \phi_{xy} - \phi_{xx} \phi_{yz}) \right. \]
\[ \left. + \phi_y^2 (\phi_{xx} \phi_{zz} - \phi_{xz}^2) + 2\phi_x \phi_z (\phi_{yz} \phi_{xy} - \phi_{yy} \phi_{xz}) \right. \]
\[ \left. + \phi_z^2 (\phi_{xx} \phi_{yy} - \phi_{xy}^2) + 2\phi_x \phi_y (\phi_{xz} \phi_{yz} - \phi_{zz} \phi_{xy}) \right] \]

where \( h = \phi_x^2 + \phi_y^2 + \phi_z^2 \)

\[ \kappa_{1,2} = H \pm \sqrt{H^2 - K} \quad \text{Principal curvatures} \]
3D Isosurface Curvature
(of a 3D level set of a signed distance function)

Again, much easier for a signed distance function $S$:

$$H = \frac{\nabla^2 S}{2} = \frac{S_{xx} + S_{yy} + S_{zz}}{2}$$

Mean curvature

$$K = \begin{vmatrix}
S_{xx} & S_{xy} \\
S_{yx} & S_{yy}
\end{vmatrix} + \begin{vmatrix}
S_{xx} & S_{xz} \\
S_{zx} & S_{zz}
\end{vmatrix} + \begin{vmatrix}
S_{yy} & S_{yz} \\
S_{zy} & S_{zz}
\end{vmatrix}$$

Gaussian curvature

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K}$$

Principal curvatures
Mean vs. Gaussian Curvature

- Zero mean curvature
  - Principal curvatures are opposite of each other ($\kappa_1 = -\kappa_2$)
  - Minimal surface (minimal surface area, like a soap film)

- Zero Gaussian curvature
  - One (or both) principal curvatures is 0 ($\kappa_1 \kappa_2 = 0$)
  - Developable surface (can be flattened onto plane without stretching)
Clinical Features Based on Curvature: Shape Index and Curvedness

\[
SI = \frac{1}{2} - \frac{1}{\pi} \arctan \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \quad (\kappa_1 \geq \kappa_2 \text{ and } 0 \leq SI \leq 1)
\]

\[
CV = \sqrt{\frac{\kappa_1^2 + \kappa_2^2}{2}} \quad (0 \leq CV < \infty)
\]

Nappi et al, Med Phys 2003
Smoothing Segmented Regions (before feature calculation)
Common Problems with Tessellated Meshes

- As produced by algorithms such as Marching Cubes
  - Common for patient-specific anatomy
  - Might want to do more than just display these surfaces with shaded surface display
  - Analysis of surface shape to provide image features

- Common problems with meshes
  - Rough surface
  - Too many triangles
  - Highly unequal edge lengths, areas, angles (i.e., sliver triangles)
Mesh Decimation

Vertex Removal
\[ V \downarrow 1 \quad T \downarrow 2 \]

Edge Collapse
\[ V \downarrow 1 \quad T \downarrow 2 \]

Half Edge Collapse
\[ V \downarrow 1 \quad T \downarrow 2 \]

(V=vertices  T=triangles)

Goal is to decrease mesh complexity while:
- Preserving overall topology
- Minimizing shape change
  - cost functions such as distance-to-plane and curvature

Mesh Decimation Example
Laplacian Mesh Smoothing
(aka isotropic diffusion)

Informally, diffusion is the spreading out of high (or low) concentrations of stuff toward the level of neighbors

\[
\frac{\partial I}{\partial t} = c \nabla^2 I = c \sum_i \frac{\partial^2 I}{\partial x_i^2}
\]

\(\nabla^2\) is the Laplacian operator:

\[
\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} + \frac{\partial^2 I}{\partial z^2}
\]

in 1D:

\[
\frac{\partial^2 I}{\partial x^2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}
\]

(finite differences kernel)

in 2D:

\[
\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\]

Laplacian is the sum of \(n\) neighbors minus \(n \times \) the center pixel

Diffusion is iterative stepping toward the mean of the neighbors
Similarity to PM Anisotropic Diffusion

**Anisotropic Diffusion**

\[
\frac{\partial I(x,y,t)}{\partial t} = \frac{1}{\Delta x^2} \left[ g(|\nabla_E I|)(\nabla_E I) \right] - \frac{1}{\Delta x^2} \left[ g(|\nabla_W I|)(\nabla_W I) \right] \\
+ \frac{1}{\Delta y^2} \left[ g(|\nabla_N I|)(\nabla_N I) \right] - \frac{1}{\Delta y^2} \left[ g(|\nabla_S I|)(\nabla_S I) \right]
\]

**Isotropic Diffusion** (g=1 and assume \(\Delta x=1, \Delta y=1\))

\[
\frac{\partial I(x,y,t)}{\partial t} = \nabla_E I - \nabla_W I + \nabla_N I - \nabla_S I
\]
\[
= I(x-1,y,t) + I(x+1,y,t) + I(x,y-1,t) + I(x,y+1,t) - 4I(x,y,t)
\]
\[
= 4 \left[ \frac{I(x-1,y,t) + I(x+1,y,t) + I(x,y-1,t) + I(x,y+1,t)}{4} - I(x,y,t) \right]
\]

**Laplacian of Gaussian:**

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & -4 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]
Laplacian Mesh Smoothing

On an image grid, pixel \textit{intensity} diffusion looks like this:

\[
\frac{\partial I}{\partial t} = c \nabla^2 I = c \sum_{i \in \{E, W, N, S\}} (I_i - I)
\]

Diffusion is stepping toward the mean of the neighbors
(we are diffusing \(I\), the concentration of stuff)

On graph or mesh structures, \textit{vertex} diffusion looks like this:

\[
\frac{\partial \vec{v}}{\partial t} = c \frac{1}{n} \sum_{v_i \in \text{neighbors}(v)} (\vec{v}_i - \vec{v})
\]

Diffusion is stepping toward centroid of neighbors
(we are diffusing \(v\), the positions of the mesh vertices)
What are the drawbacks of this \textit{isotropic} diffusion process? How could we address these drawbacks?
Level Set Mean Curvature Flow

$$\frac{d\phi}{dt} = cH = c \frac{\Delta\phi}{2}$$

Mean Curvature Flow is Isotropic Diffusion Equation

Note that curvature regularization terms are built into most level set methods so they are not typically done in an explicit step.

However, smoothing after manual editing is an example.

http://www.math.utah.edu/~mayer/math/MCF/dumbbell2_js.html

http://www.polthier.info/articles/anisotropic/
Boundary and Region Features
Binarized Region:
Centroid, Area, Volume, Diameter

**centroid** = \( \frac{1}{N} \sum_{i=1}^{N} x_i \)

**center of mass** = \( \frac{1}{N} \sum_{i=1}^{N} w_i x_i \)

**area** = \( N \cdot area_{\text{pixel}} \)

**volume** = \( N \cdot volume_{\text{voxel}} \)

\[ d_{\text{area equiv}} = \sqrt{\frac{4 \cdot \text{area}}{\pi}} \]

\[ d_{\text{vol equiv}} = \sqrt{\frac{3 \cdot \text{volume}}{\pi}} \]

\[ d_{\text{surf area equiv}} = \sqrt{\frac{\text{surf area}}{\pi}} \]
Mesh:
Perimeter, Surface Area

For polygonal contours and triangular meshes

\[
\text{perimeter} = \sum_{i=0}^{N-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}
\]

\[
\text{surface area} = \frac{1}{2} \sum_{i=1}^{N} \sqrt{\begin{vmatrix}
    x_{i,1} & y_{i,1} & 1 \\
    x_{i,2} & y_{i,2} & 1 \\
    x_{i,3} & y_{i,3} & 1
\end{vmatrix}^2 + \begin{vmatrix}
    y_{i,1} & z_{i,1} & 1 \\
    y_{i,2} & z_{i,2} & 1 \\
    y_{i,3} & z_{i,3} & 1
\end{vmatrix}^2 + \begin{vmatrix}
    x_{i,1} & z_{i,1} & 1 \\
    x_{i,2} & z_{i,2} & 1 \\
    x_{i,3} & z_{i,3} & 1
\end{vmatrix}^2}
\]

*First subscript is triangle # and second subscript is vertex #*

What to do for binary images?

Perimeter of 4-connected or 8-connected path has large errors
Surface area as sum of surface voxel faces has large errors
Feret Diameter
(aka caliper diameter)

First, compute 2D convex hull as ordered list of points, \( p_i \)

\[
D_{\text{max}} = \max_{i,j} \left| p_i - p_j \right|
\]

\[
D_{\text{min}} = \min_{i} \max_{j} \left( \frac{\left| (p_{i+1} - p_i) \times (p_i - p_j) \right|}{\left| p_{i+1} - p_i \right|} \right)
\]

\[
D_{\text{mean}} = \frac{\text{perimeter}}{\pi} = \frac{1}{\pi} \sum_{i} \left| p_{i+1} - p_i \right|
\]

Distance from point to line

Follows from Cauchy’s theorem for 2D convex bodies
Shoelace Formula for Polygon Area

Signed Area of a Triangle

\[ A_{tri} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \]

\[ = \frac{1}{2} \left( x_1y_2 + x_3y_1 + x_2y_3 - x_2y_1 - x_1y_3 - x_3y_2 \right) \]

Signed Area of a Triangle (one vertex at origin)

\[ A_{tri,origin} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \]

\[ = \frac{1}{2} (x_1y_2 - x_2y_1) \]

This does generalize to 3D...

Tetrahedron Volume in 3D

\[ V_{tetrahedron} = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \]

CCW for positive signed area
Shoelace Formula for Polygon Area

Signed Area of polygon (even concave)

\[ A = \frac{1}{2} \sum_{i=0}^{N-1} (x_i y_{i+1} - x_{i+1} y_i) \]

\[ A = \frac{1}{2} \begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix} \]

Why “shoelace”?

Centroid of polygon (even concave)

\[ \text{centroid}_x = \frac{1}{6A} \sum_{i=0}^{N-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i) \]

\[ \text{centroid}_y = \frac{1}{6A} \sum_{i=0}^{N-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i) \]

How do signed areas help handle concavity correctly?
Level Set Approach to Perimeter/Surface Area and Area/Volume

\[ H(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases} \]

Heaviside Step Function

\[ \delta(x) = \frac{\partial}{\partial x} H(x) \]

Dirac Delta Function

\[ \text{area}\{\phi \leq 0\} = \int H(\phi(x, y))\,dxdy \]

(or volume)

\[ \text{length}\{\phi = 0\} = \int |\nabla H(\phi(x, y))|\,dxdy = \int \delta(x)|\nabla \phi(x, y)|\,dxdy \]

(or surface area)

\[ H_{\text{sin}}(x) = \begin{cases} 
\frac{1}{2} \left( 1 + \frac{x}{\varepsilon} + \frac{1}{\pi} \sin \left( \frac{\pi x}{\varepsilon} \right) \right) & \text{if } |x| \leq \varepsilon \\
H(x) & \text{otherwise}
\end{cases} \]

Regularized Heaviside Step Functions

\[ H_{\text{atan}}(x) = \frac{1}{2} \left( 1 + \frac{2}{\pi} \arctan \left( \frac{x}{\varepsilon} \right) \right) \]

Chan and Vese 2001
Tumor Mass Shape and Margins

Shapes
- Round
- Oval
- Lobular
- Irregular

Margins
- Circumscribed
- Microlobulated
- Obscured
- Indistinct, ill-defined
- Spiculated

Singh and Nagarajan, 2011
## Measures of Roundness

\[ 0 \leq \text{metric} \leq 1 \]

### 2D

- **Circularity**
  \[
  \text{circularity} = \frac{r_{\text{inscribed}}}{r_{\text{circumscribed}}}
  \]
  \[
  \text{circularity} = \frac{4\pi \cdot \text{area}}{\text{perimeter}^2}
  \]
- **Convexity**
  \[
  \text{convexity} = \frac{\text{perimeter}_{\text{convex hull}}}{\text{perimeter}}
  \]
- **Solidity**
  \[
  \text{solidity} = \frac{\text{area}}{\text{area}_{\text{convex hull}}}
  \]

### 3D

- **Sphericity**
  \[
  \text{sphericity} = \frac{r_{\text{inscribed}}}{r_{\text{circumscribed}}}
  \]
  \[
  \text{sphericity} = \frac{\pi^{1/3} (6 \cdot \text{volume})^{2/3}}{\text{surface area}}
  \]
- **Convexity**
  \[
  \text{convexity} = \frac{\text{surface area}_{\text{convex hull}}}{\text{surface area}}
  \]
- **Solidity**
  \[
  \text{solidity} = \frac{\text{volume}}{\text{volume}_{\text{convex hull}}}
  \]

*(the names of these metrics vary depending on who you ask)*
Shape Parameterization
Surface (2D) Parameterization

Fast marching can be run on a triangulated mesh

Regular Grid

Triangulated Mesh

Initial wavefront can be a point or a line or a region

Fastest gradient descent creates geodesic paths that are perpendicular to isocontours

Surazhky et al, SIGGRAPH 2005

Cardiac Depolarization Times
Talbot et al, Int Foc 2013
Medial Axis Transform

- MAT = set of all points with more than one closest point to the shape boundary
- MAT = set of all points where more than one grassfire front meet

in 3D
Medial Axis Transform

Methods include (1) binary morphological thinning, (2) ridges in distance transform, (3) Voronoi diagram

\[ |\nabla D| = 1 \text{ almost everywhere} \]

Ridges are abrupt directional changes in \( \nabla D \)

MAT = union of all ridge points on Euclidean distance map
Medial Axis Transform
Medical Imaging Examples

Colon  Aorta  Lungs

Paik et al., 2002
Centerline (1D) Parameterization

- Linear
  - Digestive tract
  - Spinal cord
  - Some bones
  - Ear canal/cochlea
- Branching
  - Blood vessels
  - Bronchi
Frenet-Serret Frame along Paths

\[
\begin{bmatrix}
\frac{\partial}{\partial s} \mathbf{T} \\
\frac{\partial}{\partial s} \mathbf{N} \\
\frac{\partial}{\partial s} \mathbf{B}
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
-k & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{T} \\
\mathbf{N} \\
\mathbf{B}
\end{bmatrix}
\]

\(\mathbf{T}\) is tangent \\
\(\mathbf{N}\) is normal \\
\(\mathbf{B}\) is binormal \((\mathbf{T} \times \mathbf{N})\) \\
\(s\) is arc length \\
\(\kappa\) is curvature \\
\(\tau\) is torsion
Path Tortuosity Metrics

- **Distance Metric**
  - Path length / start-to-end length

- **Inflection Point Count**
  - Count local minima in path curvature

- **Sum of Angles Metric**
  - Integrate curvature along path and normalize by length
Cobb Angle of Spine for Scoliosis

L-R conv Max search

Edge Detection

Adaptive Threshold

ROIs

Minimum Bounding Rectangle

Cobb Angle

\[ \varphi = \max \left\{ \tan^{-1} \left( \frac{m_i - m_j}{1 + m_i \times m_j} \right) \right\} \]

Horng et al. 2019
Maximally Inscribed Spheres

Maximally Inscribed Spheres along Medial Axis to quantify atherosclerosis

Maximally Inscribed Spheres Along Medial Axis to Quantify colon distension

Hung et al, Radiology 2002
Thomas et al, Stoke 2005
Unfolding Anatomic Surfaces

Van Essen and Drury, J Neurosci 1997
Wang et al, SPIE Med Imag 2004
Yao et al, MICAI 2012
Zhu et al, IEEE TMI 2005
Paik et al, 2002
What does it mean for me?

• Methods:
  • Local Pointwise Features
  • Morphological Analysis
  • Shape Features
  • Shape Parameterization
• Many features to describe shape and geometry
• Considering natural parameterization of anatomy can be very useful

Next Lecture:
Texture Analysis